4.11 Exponential Functions

***Laws of Exponents*:** If s, t, a, and b are real numbers with $a > 0$ and $b > 0$,

then $a^s \cdot a^t = a^{s+t}$ $\quad (a^s)^t = a^{st}$ $\quad (ab)^s = a^s b^s$ $\quad 1^s = 1$ $\quad a^{-s} = \dfrac{1}{a^s} = \left(\dfrac{1}{a}\right)^s$ $\quad a^0 = 1$

An ***exponential function*** is a function of the form $f(x) = a^x$, where a is a positive real number $(a > 0)$ and $a \neq 1$. The domain of f is the set of all real numbers.

Theorem: For an exponential function $f(x) = a^x$, $a > 0$, $a \neq 1$, if x is any real number, then

$$\frac{f(x+1)}{f(x)} = a$$

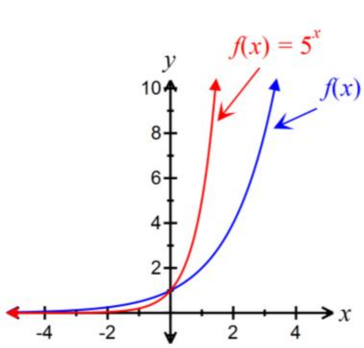
Examples: Determine whether the given functions are exponential or not.

x	$f(x)$
-1	2
0	5
1	8
2	11
3	14

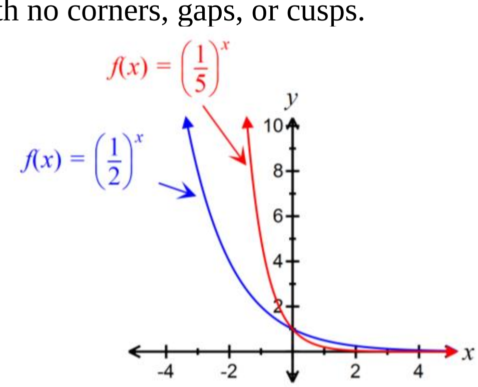
x	$f(x)$
-1	2/3
0	1
1	3/2
2	9/4
3	27/8

Properties of the Exponential Function $f(x) = a^x$, $a > 0$, $a \neq 1$

- Domain: $(-\infty, \infty)$; Range: $(0, \infty)$
- There are no x-intercepts; the y-intercept is 1.
- The x-axis $(y = 0)$ is a horizontal asymptote.
  - For $a > 1$, the graph approaches the x-axis as $x \to -\infty$.
  - For $0 < a < 1$, the graph approaches the x-axis as $x \to \infty$.
- $f(x) = a^x$ is one-to-one.
  - For $a > 1$, $f(x) = a^x$ is an increasing function.
  - For $0 < a < 1$, $f(x) = a^x$ is a decreasing function.
- The graph of f contains the points $\left(-1, \frac{1}{a}\right)$, $(0,1)$, and $(1, a)$.
- The graph of f is smooth and continuous, with no corners, gaps, or cusps.



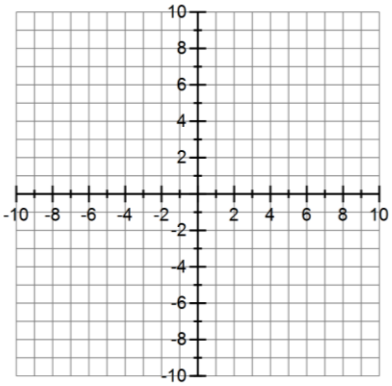
$a > 1$



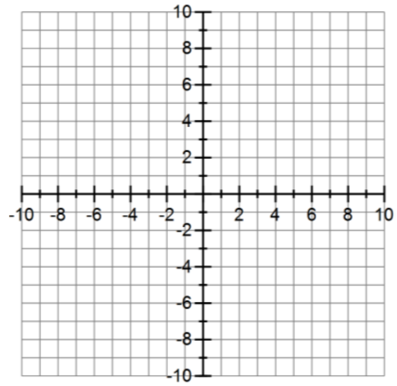
$0 < a < 1$

Examples:

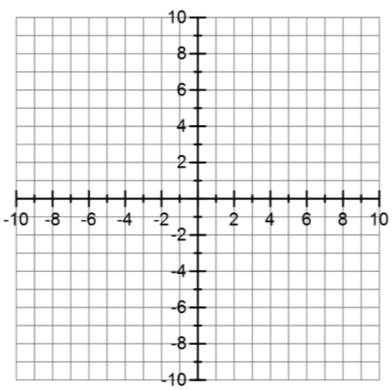
a) Graph $f(x) = 3^x$.



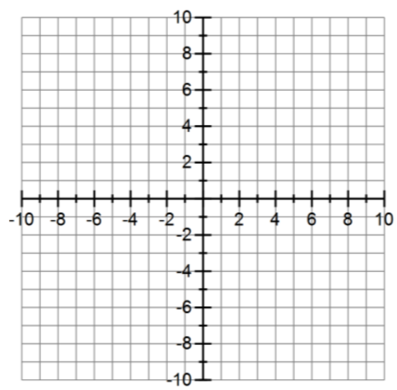
b) Graph $f(x) = 2 \cdot \left(\frac{1}{3}\right)^x$.



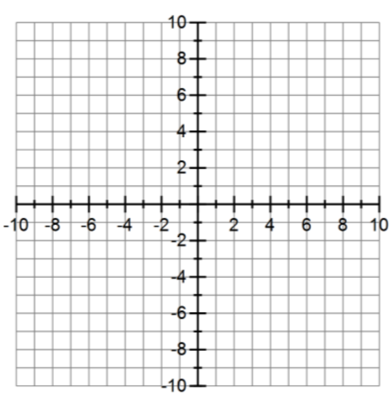
c) Graph $f(x) = 5^{x+3}$.



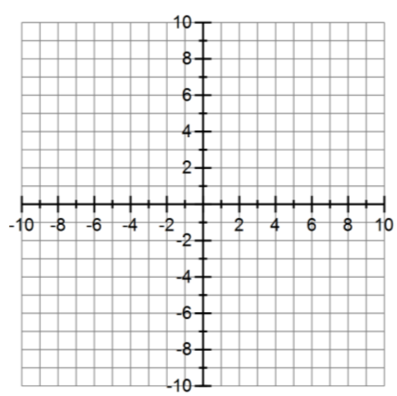
d) Graph $f(x) = \left(\frac{1}{2}\right)^x + 3$.



e) Graph $f(x) = 2^{-x}$.



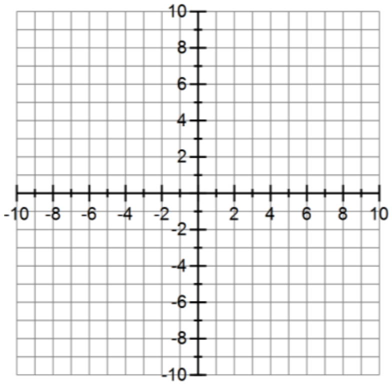
f) Graph $f(x) = -3^x$



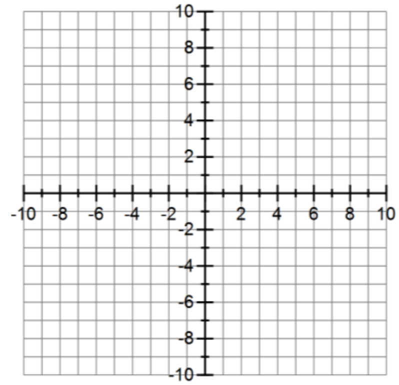
The **number *e*** (approximately 2.71828…) is defined as the number that the expression $\left(1+\frac{1}{n}\right)^n$ approaches as $n \to \infty$. In calculus, this is expressed using limit notation as $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Examples:

a) Graph $f(x) = e^x$.



b) Graph $f(x) = -e^{-x}$



Solving Exponential Equations

If $a > 0$ and $a \neq 1$ and $a^u = a^v$, then $u = v$.

Many exponential equations can be rewritten so the two sides have a common base. This allows us to set the exponents equal to each other and solve the equation.

Examples: Solve the following equations.

a) $3^{-x} = 243$

b) $5^{x+3} = \frac{1}{5}$

c) $4^{x^2} = 2^x$

d) $3^{x^2-5x} = \frac{1}{81}$

e) $4^x \cdot 2^{x^2} = 16^2$

f) $e^{x^2} = e^{3x} \cdot \frac{1}{e^2}$